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ELASTIC WAVE POLARIZATION AND VELOCITY IN MATERIALS WITH STRESS-INDUCED ANISOTROPY

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Among experimental methods for studying the stressed state of natural and artificial materials considerable attention [1-17] is devoted to a group of methods based seismo- and acoustoelastic effects. The first attempts to formulate a common rigorous theory for acoustic wave propagation in an elastic material under pressure were made in the 1940s [1, 2]. In [3] expressions were obtained for the velocities of P- and S-waves propagating in a deformable isotropic body along and across a uniaxial load. Here even for small values of prior stresses it appeared necessary to consider the elasticity moduli of not only the second, but also the third order [4]. Nonlinear elasticity theory [4] for finite strains is taken as a basis by other researchers [5-17] analyzing the effect of static stresses on elastic wave velocity.

However, in spite of the considerable number of publications use of the polarization-kinematic characteristics of elastic waves for estimating stresses is still only in the initial stage. On the basis of the disturbance method [18, 19] approximate analytical expressions are obtained in this work for characteristics of phase velocities and polarization vectors for quasi-longitudinal and quasi-transverse waves propagating in materials with stress-induced anisotropy. Stress-induced anisotropy is understood here in a narrow sense as the capacity of elastic isotropic materials to acquire anisotropy of acoustic properties with deformation, i.e., it is assumed that with zero strains anisotropy caused by the structural and substance composition may be ignored.

The approximation equations obtained are required in estimating parameters of a complex stressed state and third order elasticity moduli. The possibility of using them is evaluated by comparing calculated results for precise and approximation equations.

1. Piezoacoustic Tensor and Piezoacoustic Constants. Phase velocities v and polarization vectors $\mathbf{a} = (a_l)$ for qP- and qS-waves are found as is known [11] by determining the natural values and natural vectors of the acoustic tensor L_{il}:

$$L_{il}a_l - v^2 a_i = 0, \quad L_{il} = c_{ij,kl} n_j n_k \tag{1.1}$$

(**n** is the unit vector of the normal to the wave front). In a loaded material the elastic constants $c_{ij,kl}$ mean effective elastic constants which are connected with the stress tensor t_{jk} and with the tensor of piezoacoustic moduli $c_{mn,pq}$ by well-known relationships [11]

$$\begin{gathered} c_{ij,kl} = S_{ij,kl} + \delta_{il} t_{jk}, \quad t_{jk} = c_{mn,pq}\varepsilon_{mn}\frac{\partial x_j}{\partial \xi_p}\frac{\partial x_k}{\partial \xi_q}, \\ S_{ij,kl} = c_{mn,pq}\frac{\partial x_i}{\partial \xi_m}\frac{\partial x_j}{\partial \xi_n}\frac{\partial x_k}{\partial \xi_p}\frac{\partial x_l}{\partial \xi_q}, \quad \varepsilon_{mn} = \frac{1}{2}\left(\frac{\partial x_p}{\partial \xi_m}\frac{\partial x_p}{\partial \xi_n} - \delta_{mn}\right), \end{gathered} \tag{1.2}$$

where x_k are Lagrangian coordinates of the material subject to a complex-stressed state; ξ_k are Lagrangian coordinates of unstressed material; ε_{mn} are strain tensor components; δ_{ik} is Kronecker symbol. Here and subsequently values of elastic constants $c_{ij,kl}$, $S_{ij,kl}$, $c_{mn,pq}$ and stress j_{ik} are normalized by the value of bulk density.

Subsequently in order to distinguish the effect of a complex stressed state t_{jk} on hydrostatic pressure $(\sigma\delta_{jk})$ we shall present all tensors in the form of the sum of two tensors one of which depends on the spherical part $(\varepsilon/3)\delta_{jk}$, and the other depends on the strain tensor deviator e_{jk}:

$$\begin{gathered} \varepsilon_{jk} = (\varepsilon/3)\delta_{jk} + e_{jk}, \quad e = e_{11} + e_{22} + e_{33} = 0, \\ t_{jk} = \sigma_{jk}(\varepsilon) + \tau_{jk}(e_{jk}). \end{gathered} \tag{1.3}$$

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TABLE 1

Material	λ	μ	ρ_0, g/cm^3	α_P	α_S	β_P	β_S
	GPa						
Granite (dry) [15]	21,16	18,38	2,65	−3546	−1219	−706,5	−42,34
Granite (moist) [16]	29,7	25,3	2,66	−1330	−267,6	−656,7	−123,0
Polystyrene [3]	2,89	1,38	1,06	−34,64	−4,47	−9,65	−0,41
Pyrex glass [3]	13,5	27,5	—	25,81	8,52	12,67	2,41
Armco-iron [3]	110	82	—	−28,05	−17,11	−17,44	2,18
Iron [6]	113	81	—	−14,49	−4,16	−10,85	−1,80
Copper [6]	105	47	—	−15,91	−3,27	−16,40	−3,65
Steel [17]	115,8	79,8	—	−14,03	−4,06	−7,71	−0,62
Water−oil emulsion [20]	1,56	0	—	−1556	0	0	0

Proceeding from the representation given by Murnaghan [4] as a cubic form of the elastic potential Σ of an isotropic material:

$$\Sigma = -p_0\varepsilon + \frac{\lambda + 2\mu}{2}\varepsilon^2 - 2\mu d + \frac{l + 2m}{3}\varepsilon^3 - 2m\varepsilon d + nD, \tag{1.4}$$

where p_0 is the initial hydrostatic pressure corresponding to zero strains; λ, μ are Lamé constants; l, m, n are Murnaghan constants relating to the marked nonlinear nature of the dependence of stress on strains; ε, d, D are three strain tensor invariants:

$$\begin{gathered} \varepsilon = Sp(\varepsilon_{jk}) = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}, \\ d = \Big(\varepsilon^2 - \sum_{j,k=1}^{3} \varepsilon_{jk}^2\Big) = \varepsilon_{11}\varepsilon_{22} + \varepsilon_{22}\varepsilon_{33} + \varepsilon_{33}\varepsilon_{11} - \varepsilon_{12}^2 - \varepsilon_{13}^2 - \varepsilon_{23}^2, \\ D = \det(\varepsilon_{jk}) = \varepsilon_{11}\varepsilon_{22}\varepsilon_{33} + 2\varepsilon_{12}\varepsilon_{13}\varepsilon_{23} - \varepsilon_{11}\varepsilon_{23}^2 - \varepsilon_{22}\varepsilon_{13}^2 - \varepsilon_{33}\varepsilon_{12}^2, \end{gathered} \tag{1.5}$$

for piezoacoustic moduli $c_{mn,pq}$ taking account of Eqs. (1.2)-(1.5) it is possible to obtain the following expression

$$\begin{gathered} c_{mn,pq} = \frac{\partial^2\Sigma}{\partial\varepsilon_{mn}\partial\varepsilon_{pq}} = \Lambda_{mn,pq} + M_{mn,pq}(\varepsilon_{jk}), \\ \Lambda_{mn,pq} = \lambda\delta_{mn}\delta_{pq} + \mu(\delta_{mp}\delta_{nq} + \delta_{mq}\delta_{np}), \\ M_{mn,pq}(\varepsilon_{jk}) = 2(l - m/3 + n/6)\varepsilon\delta_{mn}\delta_{pq} + (m - n/6)\varepsilon(\delta_{mp}\delta_{nq} + \delta_{mq}\delta_{np}) + \\ (2m - n)(\delta_{mn}e_{pq} + \delta_{pq}e_{mn}) + \\ (n/2)(\delta_{mp}e_{nq} + \delta_{nq}e_{mp} + \delta_{mq}e_{np} + \delta_{np}e_{mq}). \end{gathered} \tag{1.6}$$

Now we shall assume that the previous stressed state with a sufficient degree of accuracy is only described by linear terms of the strain tensor; then after substituting Eqs. (1.6) in Eqs. (1.2) stress tensor t_{jk} and tensor $S_{ij,kl}$ are written in the form

$$\begin{gathered} t_{jk} = \Lambda_{jk,mn}\varepsilon_{mn};\ S_{ij,kl} = S^0_{ij,kl} + S^e_{ij,kl}, \\ S^0_{ij,kl} = [\lambda + 2(2\lambda/3 + l - m/3 + n/6)\varepsilon]\delta_{ij}\delta_{kl} + \\ [\mu + (4\mu/3 + m - n/6)\varepsilon](\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \\ S^e_{ij,kl} = (2\lambda + 2m - n)(\delta_{ij}e_{kl} + \delta_{kl}e_{ij}) + \\ (2\mu + n/2)(\delta_{ik}e_{jl} + \delta_{jl}e_{ik} + \delta_{il}e_{jk} + \delta_{jk}e_{il}). \end{gathered} \tag{1.7}$$

By placing Eqs. (1.7) in Eq. (1.1) for the piezoacoustic tensor L_{il} we have

$$\begin{gathered} L_{il} = L^0_{il} + L^e_{il}, \\ L^0_{il} = [\mu + (\lambda + 2\mu + m - n/6)\varepsilon]\delta_{il} + \\ [\lambda + \mu + (4\lambda + 4\mu + 6l + m + n/2)\varepsilon/3]n_i n_l, \\ L^e_{il} = (2\mu + n/2)e_{il} + (4\mu + n/2)\delta_{il}(e_{jk}n_j n_k) + \\ 2(\lambda + \mu + m - n/4)(n_i n_k e_{kl} + e_{ik}n_k n_l). \end{gathered} \tag{1.8}$$

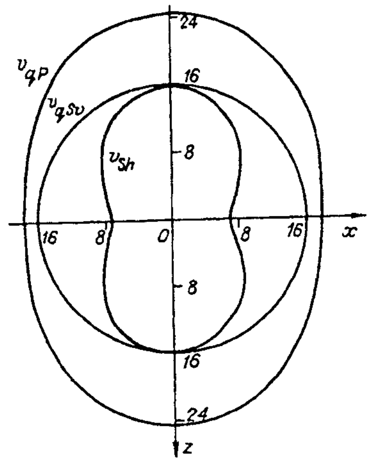

Fig. 1

It is noted that the first term describes the change in acoustic tensor due to deformation of the volume, and the second defines development of acoustic anisotropy with shape deformation. On the basis of Hooke's law (1.7) dilation ε and the strain deviator are connected with stress tensor by simple relationships

$$t_{jk} = \sigma\delta_{jk} + \tau_{jk},\ \sigma = (\lambda + 2/3\mu)\varepsilon,\ \tau_{jk} = 2\mu e_{jk}. \tag{1.9}$$

By substituting Eq. (1.9) in Eqs. (1.8) for piezoacoustic tensor L_{il} we obtain

$$\begin{aligned} L^0_{il} &= [\mu + \alpha_S\sigma]\delta_{il} + [\lambda + \mu + (\alpha_P - \alpha_S)\sigma]n_in_l, \\ L^e_{il} &= 2\beta_S\tau_{il} + (1 + 2\beta_S)\delta_{il}(\tau_{jk}n_jn_k) + \\ &\quad (\beta_P - 1 - 4\beta_S)(n_in_k\tau_{kl} + \tau_{ik}n_kn_l)/2, \end{aligned} \tag{1.10}$$

where α_P, α_S, β_P, β_S are piezoacoustic constants:

$$\begin{aligned} \alpha_P &= (7\lambda + 10\mu + 6l + 4m)/(3\lambda + 2\mu), \\ \beta_P &= (2\lambda + 5\mu + 2m)/\mu, \\ \alpha_S &= (3\lambda + 6\mu + 3m - n/2)/(3\lambda + 2\mu), \\ \beta_S &= (4\mu + n)/(8\mu). \end{aligned} \tag{1.11}$$

It is noted that $\beta_P = 2[\alpha_S(3\lambda + 2\mu)/\mu + 4\beta_S - 1/2]/3$. Given in Table 1 are values of piezoacoustic constants for some materials (1.11) calculated for an experimental study of elastic wave velocities with all-round and uniaxial loading [3, 6, 15-17, 20].

2. Velocities and Polarization Vectors of qP- and qS-Waves with Triaxial Loading. In this case phase velocities and polarization vectors of qP- and qS-waves depend in a very complex way on the strain tensor deviator. It is assumed that shape deformation compared with overall deformation of the volume is a higher order effect (i.e., $\varepsilon \gg e_{jk}$ and $L_{il}^0 \gg L_{il}^e$), then in order to obtain approximate analytical expressions for phase velocities and polarization vectors of qP- (v_3, a_{i3}) and qS-waves (v_1, a_{i1}), (v_2, a_{i2}) it is possible to use a linear approximation of the disturbance method [18, 19]:

$$v_3^2 = D_{33},\ v_{1,2}^2 = (D_{11} + D_{22} \pm \sqrt{(D_{22} - D_{11})^2 + 4(D_{12})^2})/2; \tag{2.1}$$

$$\begin{aligned} a_{i3} &= b_{i3} + \Delta(b_{i2}\sin\Gamma + b_{i1}\cos\Gamma), \\ a_{i2} &= -\Delta\sin(\Gamma - \gamma)b_{i3} + \{b_{i2}\cos(\hat\Gamma - \gamma) + b_{i1}\sin(\hat\Gamma - \gamma)\}/\cos\hat\Gamma, \\ a_{i1} &= -\Delta\cos(\Gamma - \gamma)b_{i3} + \{-b_{i2}\sin(\hat\Gamma - \gamma) + b_{i1}\cos(\hat\Gamma - \gamma)\}/\cos\hat\Gamma; \end{aligned} \tag{2.2}$$

$$\mathrm{tg}\,\Gamma = D_{23}/D_{13},\ \Delta = \sqrt{D_{23}^2 + D_{13}^2}/(\hat{v}_P^2 - \hat{v}_S^2),$$
$$\mathrm{tg}\,2\gamma = 2D_{12}/(D_{11} - D_{22}),\ \mathrm{tg}\,\hat{\Gamma} = -\Delta^2 \sin 2(\Gamma - \gamma)/(2\Omega), \tag{2.3}$$
$$\Omega = \sqrt{(D_{22} - D_{11})^2 + 4(D_{12})^2}/(\hat{v}_P^2 - \hat{v}_S^2);$$

$$D_{nr} = \sum_{i,k=1}^{3} b_{in} L_{ik} b_{kr} = \sum_{i,j,k,l=1}^{3} \hat{c}_{ij,kl} b_{in} n_j n_l b_{kr}. \tag{2.4}$$

Here $\hat{v}_S^2 = \mu + \alpha_S\sigma$, $\hat{v}_P^2 = \lambda + 2\mu + \alpha_P\sigma$ are characteristic values; b_1, b_2, b_3 are characteristic vectors of the piezoacoustic tensor L_{il}^0 (phase velocities and polarization vectors of SV- and SH-, and P-waves in isotropic material):

$$b_{i1} = (\cos\varphi\cos\theta,\ \sin\varphi\cos\theta, -\sin\theta),$$
$$b_{i2} = (-\sin\varphi, \cos\varphi, 0), \tag{2.5}$$
$$b_{i3} = n_i = (\cos\varphi\sin\theta,\ \sin\varphi\sin\theta,\ \cos\theta).$$

In order to obtain approximate analytical expressions for the transit time of qP- and qS-waves in uniform material it is also possible to use the disturbance method [18, 19]:

$$t_r(\mathrm{l}) = \sqrt{l_x^2 + l_y^2 + l_z^2}/v_r(\varphi,\theta),\ \mathrm{tg}\,\varphi = l_y/l_x,\ \mathrm{tg}\,\theta = \sqrt{l_x^2 + l_y^2}/l_z \tag{2.6}$$

($\mathbf{l} = (l_x, l_y, l_z)$ is the vector connecting the source and receiver). By substituting Eqs. (2.5) in Eq. (2.4), for elements of matrix D_{nr} we obtain

$$D_{11} = \hat{v}_S^2 + 2\beta_S T_{11} + (1 + 2\beta_S)T_{33},\ D_{12} = 2\beta_S T_{12},$$
$$D_{22} = \hat{v}_S^2 + 2\beta_S T_{22} + (1 + 2\beta_S)T_{33},\ D_{23} = ((\beta_P - 1)/2)T_{23}, \tag{2.7}$$
$$D_{33} = \hat{v}_P^2 + \beta_P T_{33},\ D_{13} = ((\beta_P - 1)/2)T_{13},\ T_{nr} = \sum_{i,k=1}^{3} b_{in}\tau_{ik}b_{kr}.$$

Thus, on the basis of Eqs. (2.1)-(2.3), (2.7) approximate expressions for phase velocities and polarization vectors may be written in the form

$$v_3^2(\mathrm{n}|\tau_{ik}) = \hat{v}_P^2 + \beta_P\sigma_{nn},\ \sigma_{nn} = T_{33},\ v_{1,2}^2(\mathrm{n}|\tau_{ij}) =$$
$$\hat{v}_S^2 + \sigma_{nn} - \beta_S\sigma_{n_kn_k},\sigma_{n_kn_k} = (T_{11} + T_{22} \pm \sqrt{(T_{11} - T_{22})^2 + 4T_{12}^2}),$$
$$\mathrm{tg}\,\Gamma = T_{23}/T_{13},\ \mathrm{tg}\,2\gamma = 2T_{12}/(T_{11} - T_{22}), \tag{2.8}$$
$$\Delta = [(\beta_P - 1)/(\hat{v}_P^2 - \hat{v}_S^2)]\eta/2,\ \eta = \sqrt{T_{13}^2 + T_{23}^2},$$
$$\mathrm{tg}\,\hat{\Gamma} = -\Delta^2\sin 2(\gamma - \Gamma)/(2\Omega),$$
$$\Omega = [2\beta_S/(\hat{v}_P^2 - \hat{v}_S^2)]\rho,\ \rho = \sqrt{(T_{11} - T_{22})^2 + 4T_{12}^2},$$

where $v_3(n)$ and $v_{1,2}(n)$ are characteristics of the velocities of qP- and qS-waves in loaded material; v_P, v_S are velocities of P- and S-waves depending on the magnitude of hydrostatic pressure; σ_{nn} is normal deviator stress operating on an area perpendicular to the propagation direction **n**; $\sigma_{n_1n_1}$ and $\sigma_{n_2n_2}$ are maximum and minimum normal deviator stresses operating on areas with normals perpendicular to the propagation direction **n**; η is tangential deviator stress operating on an area perpendicular to the propagation direction **n**; Γ is tangential stress azimuth; ρ is maximum tangential deviator stress operating on areas with normals perpendicular to the propagation direction **n**; γ is azimuth of this tangential stress. It is noted that if $\Omega = 0$, then $\gamma = \Gamma$ and $\hat{\Gamma} = 0$. With known λ and μ for longitudinal wave velocities it is only possible to determine two third order elasticity moduli: l and m, and for transverse wave data it is only possible to determine two: m and n.

On the basis of the expression obtained for piezoacoustic tensor (1.10) values are calculated for the phase velocities of quasi-longitudinal and quasi-transverse waves (Fig. 1) in a granite specimen with parameters $v_P = 5.5$ km/sec, $v_S = 3.1$ km/sec, $\rho_0 = 2.66$ g/cm^3, $\alpha_P = -1330$, $\alpha_S = -268$, $\beta_P = -657$, β_S -123 with biaxial loading (axial pressure $\sigma_0 = 40$ MPa, side pressure $\sigma_1 = 20$ MPa), and presented in Fig. 2 are the errors in calculating velocities by approximation Eqs. (2.8) and deviation of the polarization vector for a qP-wave in the propagation direction. Whence it is possible to conclude that with biaxial loading approximation expression (2.8) for a qSh-wave agrees with the accurate expression (a qSh-wave is a purely transverse Sh-wave), and for qP- and qSv-waves the approximation expressions coincide with accurate expressions in the

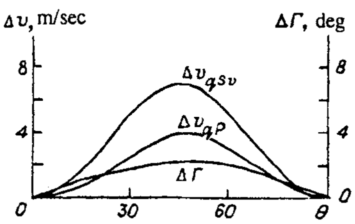

Fig. 3

directions of the principal loading axes since in these directions they propagate purely longitudinally and purely transverse waves; the maximum errors in the calculations by approximation equations arise in directions which are 45° from the directions of the principal loading axes. However, absolute values of the errors (Fig. 2) even with quite strong deviation of the stressed state from all-round compression (σ_0 = 40 MPa, σ_1 = 20 MPa) does not exceed 4 m/sec for qP- and 7 m/sec for qSv-waves. Whence it is possible to see that the approximation obtained is quite accurate.

In order to estimate the intensity of deviator stresses τ_{ij} relative to normal stress τ_{33} operating on a horizontal area, for this study of velocities (2.8) it is possible to use an equation

$$\tau_{ij} = \tau_{33}\frac{3W_{ij} - \delta_{ij}(W_{11} + W_{22} + W_{33})}{2W_{33} - W_{11} - W_{22}},$$

where W_{ij} are parameters of the elliptical approximation, characteristic of qP-wave velocity v_3, or average velocities of qS-waves $(v_1 + v_2)/2$. An estimate of absolute stress values may be found by additional measurements of normal stress τ_{33} or piezoacoustic constant β.

Thus, the relationship obtained in this work may serve as a theoretical basis for developing seismoacoustic methods in studying the complex stressed state of elastic materials.

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